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Also resolving perpendicular to GL

$$0 = u \sin \phi \cdot t - \tfrac{1}{2} g \cos \beta \cdot t^2. \tag{2}$$

Hence from (1) and (2) $GL = [2u^2 \cos(\phi + \beta) \sin \phi]/[g \cos^2 \beta]$ when the expression on the right-hand side is a maximum, *i.e.*, $GL = u^2[\sin(2\phi + \beta) - \sin \beta]/[g \cos^2 \beta]$; whence $2\phi + \beta = \pi/2$, and

$$GL = \frac{u^2(1 - \sin \beta)}{g \cos^2 \beta} = \frac{u^2}{g(1 + \sin \beta)}. \tag{3}$$

Again, if LE is drawn perpendicular to GA, and GF is the horizontal projection of GE, we have

$$\sin \beta = \frac{NL}{GL} = \frac{NL}{GE} \cdot \frac{GE}{GL} = \frac{FE}{GE} \cdot \frac{GE}{GL} = \sin \alpha \cdot \cos \theta.$$

Hence, (3) becomes $(u^2/g)/GL = 1 + \sin \alpha \cos \theta$, *i.e.*, L is on an ellipse, focus G, eccentricity $\sin \alpha$, and the semi-latus rectum u^2/g.

REMARK BY OTTO DUNKEL, Washington University—This problem may also be solved by finding the intersection of the envelope of the trajectories (see Granville, *Calculus*, first edition, p. 216) $z = (u^2/2g) - (g/2u^2)(x^2 + y^2)$ and the inclined plane $z = y \tan \alpha$.

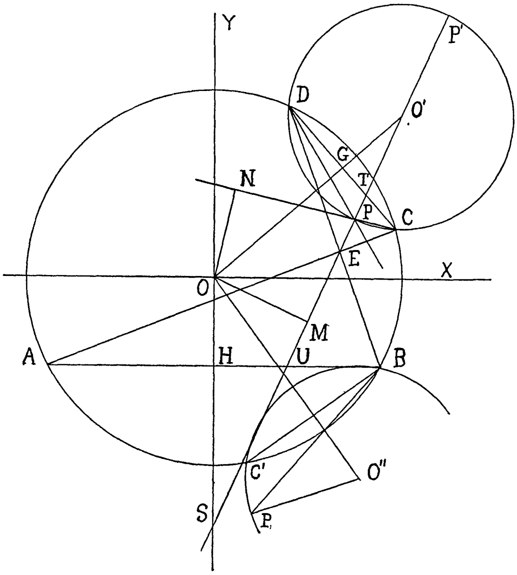
Also solved by AUGUSTUS BOGARD, A. M. HARDING, and WILLIAM HOOVER.

2865 [1920, 482]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

In a circle, a chord AB is fixed in position and a moving chord CD is constant in length. Find the locus of the intersection of the bisectors of the angles ACD and BDC.

I. SOLUTION BY F. L. WILMER, Omaha, Neb.

A solution of the problem in all its ramifications requires a discussion of a number of distinct cases, segregable into two classes according as the lines connecting the extremities of the two chords do or do not cross within the given circle. One special case of the first class has been selected here for discussion to show a workable method of attack of the problem in the various cases.



Let the radius of the given circle be the unit of length and suppose that the internal bisectors of the angles of the triangle CDE (see figure) meet in P, a point of the locus. Let the center O of the unit circle be the origin of rectangular coördinates with the x-axis parallel to AB. It will be seen at once that the acute angle b which PC makes with DP is constant, so that P, C, D, and P' (the intersection of the two external bisectors of the angles C and D) lie upon a circle of constant radius with center at O'. The central angle $DO'C = 2b$. Denote the lengths of the arcs AB and CD of the fixed circle by c and a, the angle YOO' by φ. When $\varphi = 0$, P lies on the y-axis and the angle $OO'P$ is also zero. When CD takes the position $C'D'$, $D' = B$, $\angle\, OO'P$ becomes in the new position at O'' $\angle\, OO''P_1 = \angle OO''C' + \angle\, C'O''P_1 = b + 2\angle\, C'D'P_1 = b + a/2$. The angle φ is now seen from the figure to be $\angle\, YOO'' = \angle\, YOB + a/2 = \pi - c/2 + a/2 = 2b + a$, since $a + c + 4b = 2\pi$. Hence $\angle\, YOO'' = 2\angle\, OO''P_1$ and it may be shown that for any position of P this relation is true. Hence $\angle\, OO'P = \varphi/2$.

We shall obtain the equations of PC and $O'P$ using φ as a parameter. Let ON and OM be the perpendiculars from the center O upon PC and $O'P$, respectively. Since $\angle NOO' = \angle NCD = 1/2 \angle DO'P = b/2 + \varphi/4$, $\angle YON = \varphi - b/2 - \varphi/4 = 3\varphi/4 - b/2$; also $\angle YOM = \pi/2 + \varphi/2$ and $\angle NOC = \angle NOO' + a/2 = (a + b)/2 + \varphi/4$. Then $ON = \cos [(a + b)/2 + \varphi/4]$ and $OM = r \sin (\varphi/2)$, where $r = OO'$. Hence the coördinates of P satisfy the equations

$$x \sin \left(\frac{3}{4}\varphi - \frac{b}{2}\right) + y \cos \left(\frac{3}{4}\varphi - \frac{b}{2}\right) = \cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right),$$

$$x \cos \frac{\varphi}{2} - y \sin \frac{\varphi}{2} = r \sin \frac{\varphi}{2},$$

and by solving these equations we obtain the parametric equations of the locus of P:

$$x = \sin \frac{\varphi}{2} \left[\cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right) + r \cos \left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos \left(\frac{1}{4}\varphi - \frac{1}{2}b\right),$$

$$y = \left[\cos \frac{\varphi}{2} \cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right) - r \sin \frac{\varphi}{2} \sin \left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos \left(\frac{1}{4}\varphi - \frac{1}{2}b\right).$$

II. Solution by Otto Dunkel, Washington University.

Let O be the center of the given circle and let us suppose that, when CD lies on the same side of AB as O, the intersection E of DB and CA lies within the circle, and that P is the intersection of the internal bisectors of the angles of the triangle ECD. Indicate by G and H the middle points of the chords CD and AB, respectively, and by O' and S the points in which the bisector EP meets OG and OH. The external bisectors of the angles D and C of the triangle ECD are perpendicular, respectively, to PD and PC, and meet EP in P', the center of the escribed circle in $\angle DEC$. Hence PP' is a diameter of the circle through P, D, P', C, with O' as center. This circle has a constant radius since $\angle DEC$ is constant in magnitude and hence $\angle DPC$ has also a constant value; also OO' is constant in length. Let EP meet CD and AB in T and U, respectively; the construction of the figure shows that the triangles AUE and DTE are similar, and, therefore, that $\angle HUS = \angle GTO'$, and, finally, that $\angle HSU = \angle GO'T$. The triangle OSO' is thus isosceles and S is a fixed point. This determines an easy construction for the curve as follows: With O as center draw a fixed circle of radius OO' passing through S; draw a variable chord SO' and lay off upon it the constant lengths $O'P$ and $O'P'$; then P is the center of the inscribed circle of ECD (in the position mentioned above) and P' is the center of the escribed circle in $\angle DEC$, both of which are points of the locus. The curve is, therefore, the *limaçon of Pascal.*

When C falls on B, P also falls upon B; when D coincides with A, P also coincides with A. When CD or any part of it is on the side of AB opposite to that of O, E and the two points P and P' are all outside of the given circle, the latter two being escribed centers in $\angle CDE$ and $\angle DCE$, respectively. After one revolution of O' and CD, P and P' are interchanged.

The equation of the locus is easily obtained in polar coördinates, taking S as the pole, SO as the axis, $\angle OSP = \theta$, $SP = \rho$. It will be convenient to take the radius OD of the given circle as unity and to denote the lengths of the arcs CD and AB by a and c, respectively, and the acute angle which PC makes with DP by b. Then $b = 1/2 \angle CEB = \pi/2 - (a + c)/4$. In the triangle ODO', $\angle DO'O = b$ and, by the Law of Sines we have $OO' = \sin (b + a/2)/\sin b = \cos (a/4 - c/4)/\cos (a/4 + c/4)$; $O'D = \sin (a/2)/\sin b = \sin (a/2)/\cos (a/4 + c/4)$. Hence

$$\rho = 2 \frac{\cos \frac{a-c}{4}}{\cos \frac{a+c}{4}} \cos \theta - \frac{\sin \frac{a}{2}}{\cos \frac{a+c}{4}}.$$

Reversing the order of the points C and D causes E to be outside of the given circle in the initial position. In order to obtain the equation for this case we had merely to replace a by $-a$ in the above equation and in the expression for OS. The remaining cases may be treated in a similar manner.

2869 [1921, 36]. Proposed by the late L. G. WELD.

The successive segments of a broken right line are represented by the successive terms of the harmonic progression, 1, 1/2, 1/3, 1/4, *ad infinitum*. Each segment makes with the preceding a